Signals and Linear Systems

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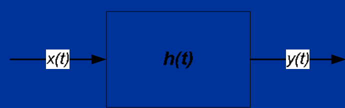
Preview

- review the linear systems analysis used in the analysis of communication systems.
  - linear time-invariant (LTI) systems
  - Frequency domain
  - Fourier series and transforms;
  - power and energy concepts,
  - the sampling theorem,
  - lowpass representation of bandpass signals.

System

- The input-output relation of a linear time-invariant system is integral defined by



$$y(t) = x(t) \star h(t)$$
$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

- h(t) denotes the impulse response of the system, x(t) is the input signal, and y (t) is the output signal

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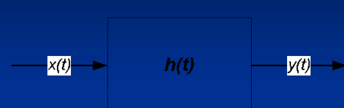
System



$$x(t) = Ae^{j2\pi f_0 t}$$

$$y(t) = \int_{-\infty}^{+\infty} Ae^{j2\pi f_0 (t-\tau)} h(\tau) d\tau$$
$$= A\left[\int_{-\infty}^{+\infty} e^{-j2\pi f_0 \tau} h(\tau) d\tau\right] e^{j2\pi f_0 t}$$
$$= \left[\int_{-\infty}^{+\infty} e^{-j2\pi f_0 \tau} h(\tau) d\tau\right] x(t)$$

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Fourier Series

- Any periodic signall $x(t)$ with period T_0 can be expressed as

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi nt/T_0}$$

- the X_n are called the Fourier series coefficients of the signal $x(t)$ and are given by

$$X_n = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi nt/T_0} dt$$

Fourier Series

- Here α is an arbitrary constant chosen in such a way that the computation of the integral is simplified.
- The frequency $f_0 = 1/T_0$ is called the fundamental frequency of the periodic signal, and the frequency $f_n = nf_0$ is called the n-th harmonic.
- In most cases either $\alpha = 0$ or $\alpha = T_0/2$ is a good choice.

Deret Fourier

- Jika $x(t)$ real, maka $x^*(t) = x(t)$
- Ganti n dengan $-n$, didapatkan $a_{-n}=a^*_n$ atau $a_n=a^*_{-n}$

$$x^*(t) = x(t) = \sum_{n=-\infty}^{+\infty} X_n^* e^{-jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_{-n}^* e^{jn\omega_0 t}$$

$$x(t) = a_0 + \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t} + X_{-n} e^{-jn\omega_0 t}$$

$$x(t) = a_0 + \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t} + X_n^* e^{-jn\omega_0 t}$$

$$\omega_0 = 2\pi / T_0$$

Deret Fourier

- Penjumlahan konjugate kompleks menghasilkan

$$x(t) = a_0 + 2\sum_{n=1}^{+\infty} \mathrm{Re}\left\{X_n e^{nk\omega_0 t}\right\}$$

- Jika $X_n = A_n \, e^{j\theta_n}$

$$x(t) = a_0 + 2\sum_{n=1}^{+\infty} A_n Cos\left(n\omega_0 t + \theta_n\right)$$

- Jika $X_n = B_n + j\, C_n$

$$x(t) = a_0 + 2\sum_{n=1}^{+\infty} \left[B_n Cos(n\omega_0 t) - C_n Sin(n\omega_0 t)\right]$$

Deret Fourier

$$x(t).e^{-jk\omega_0 t} = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}.e^{-jk\omega_0 t}$$

$$\int_0^{T_0} x(t).e^{-jk\omega_0 t} dt = \int_0^{T_0} \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}.e^{-jk\omega_0 t} dt$$

$$\int_0^{T_0} x(t).e^{-jk\omega_0 t} dt = \sum_{n=-\infty}^{+\infty} X_n \left[\int_0^{T_0} e^{j(n-k)\omega_0 t} dt \right]$$

Deret Fourier

$$\int_0^{T_0} e^{j(n-k)\omega_0 t} dt = \left[\begin{array}{l} T_0, n = k \\ 0, n \neq k \end{array} \right.$$

$$\int_0^{T_0} x(t).e^{-jk\omega_0 t} dt = \sum_{n=-\infty}^{+\infty} X_n \left[\int_0^{T_0} e^{j(n-k)\omega_0 t} dt \right]$$

$$\int_0^{T_0} x(t).e^{-jk\omega_0 t} dt = X_k .T_0$$

sehingga

$$X_k = \frac{1}{T_0} \int_0^{T_0} x(t).e^{-jk\omega_0 t} dt$$

Deret Fourier

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n . e^{jn\omega_0 t}$$

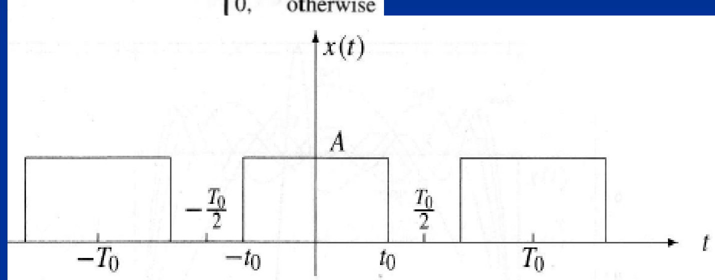
$$X_n = \frac{1}{T_0} \int_0^{T_0} x(t) . e^{-jn\omega_0 t} dt$$

- Koefisien a_k disebut koefisien deret Fourier atau koefisien spektral
- Komponen dc = X_0 :

$$X_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

Deret Fourier

$$x(t) = A\Pi\left(\frac{t}{2t_0}\right) = \begin{cases} A, & |t| < t_0 \\ \frac{A}{2}, & t = \pm t_0 \\ 0, & \text{otherwise} \end{cases}$$



- Assuming A=1,T=4, and t_0= 1 ,
  1. Determine the Fourier series coefficients of x(t) in exponential and trigonometric form.
  2. Plot the discrete spectrum of x(t).

Deret Fourier

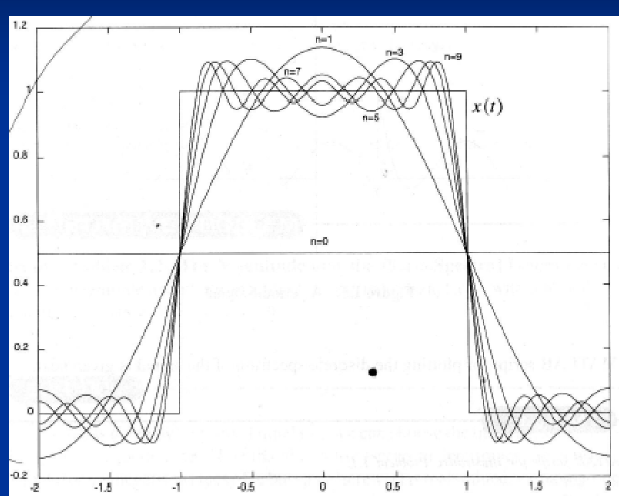
$$X_n = \frac{1}{4}\int_{-1}^{+1} e^{-j2\pi nt/4}\,dt$$

$$= \frac{1}{-2j\pi n}\left[e^{-j2\pi n/4} - e^{j2\pi n/4}\right]$$

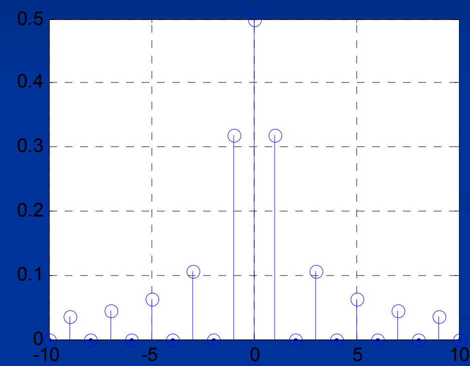
$$= \frac{1}{2}\sin c\left(\frac{\pi n}{2}\right)$$

$$\sin c(x) = \frac{\sin(x)}{x}$$

Deret Fourier



Deret Fourier



- The discrete spectrum of the signal

Deret Fourier

```
n=-10:10;
y=0.5*sinc(pi*n/2);
stem(n,y); grid on;
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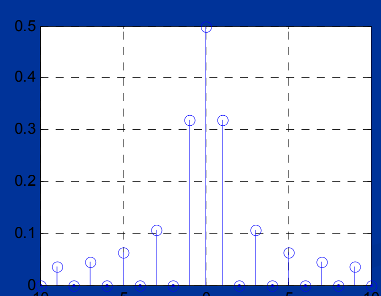
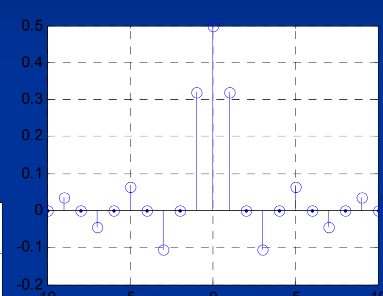
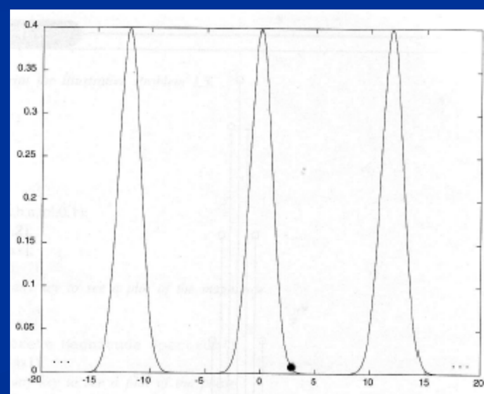




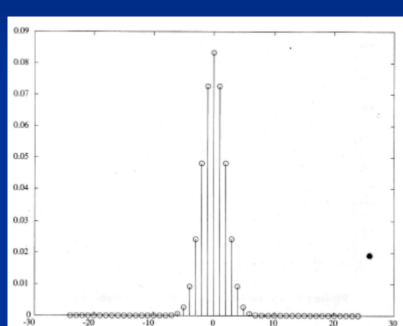
Deret Fourier

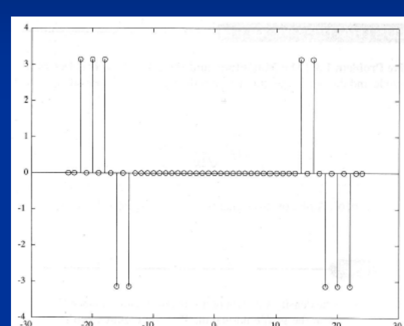
- [The Magnitude and the Phase Spectra] Determine and plot the magnitude and the phase spectra of a periodic signal with a period equal to 12 (in the interval [-6, 6]) that is given by

$$x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$



Deret Fourier





- The magnitude and phase spectra

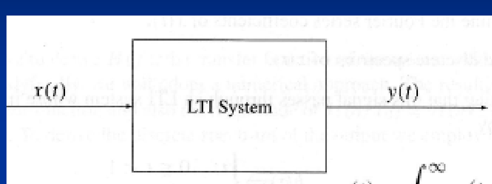
Periodic Signals and LTI Systems

- When a periodic signal x (t) is passed through a linear time-invariant (LTI) system, the output signal y(t) is also periodic,
- usually with the same period as the input signal,
- therefore it has a Fourier series expansion.

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$$

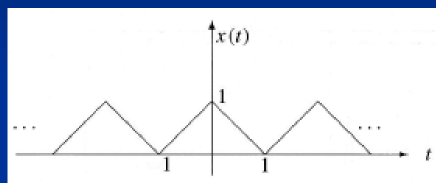
$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0}$$

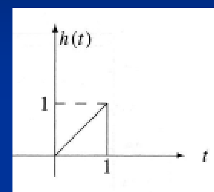
Periodic Signals and LTI Systems



$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n(t-\tau)/T_0} h(\tau)\,d\tau \\ &= \sum_{n=-\infty}^{\infty} x_n \left(\int_{-\infty}^{\infty} h(\tau) e^{-j2\pi n\tau/T_0}\,d\tau \right) e^{j2\pi nt/T_0} \\ &= \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0} \end{aligned}$$

Periodic Signals and LTI Systems





$$\Lambda(t) = \begin{cases} t+1, & -1 \le t \le 0 \\ -t+1, & 0 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

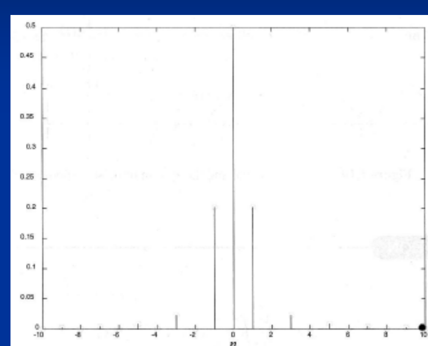
x(t)

$$h(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

h(t)

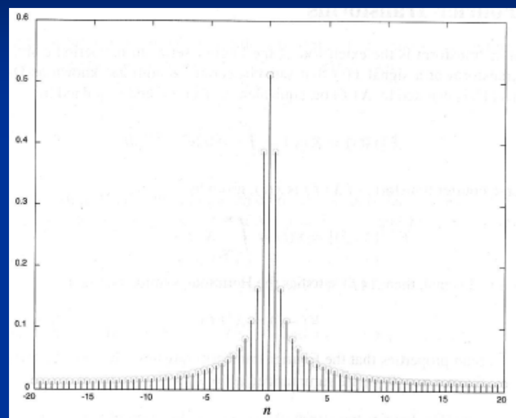
Periodic Signals and LTI Systems

$$\begin{aligned} x_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi nt/T_0}\,dt \\ &= \frac{1}{2}\int_{-1}^{1} \Lambda(t)e^{-j\pi nt}\,dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \Lambda(t)e^{-j\pi nt}\,dt \\ &= \frac{1}{2}\mathcal{F}[\Lambda(t)]_{f=n/2} \\ &= \frac{1}{2}\,\text{sinc}^2\left(\frac{n}{2}\right) \end{aligned}$$



- The discrete spectrum of the signal x(t)

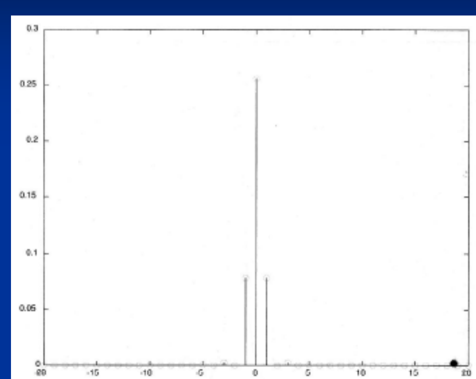
Periodic Signals and LTI Systems



- The magnitude of H (n/2)

Periodic Signals and LTI Systems

$$y_n = x_n H\left(\frac{n}{T_0}\right) = \frac{1}{2}\,\mathrm{sinc}^2\left(\frac{n}{2}\right) H\left(\frac{n}{2}\right)$$



- The discrete spectrum of the output

Fourier Transforms

$$\mathcal{F}[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

- The inverse Fourier transform of X(f) is x(t)

$$\mathcal{F}^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

- If x (t) is a real signal, then X (f) satisfies the Hermitian symmetry,

$$X(-f) = X^*(f)$$

Fourier transform properties

- Linearity

$$\mathcal{F}[\alpha x_1(t) + \beta x_2(t)] = \alpha\mathcal{F}[x_1(t)] + \beta\mathcal{F}[x_2(t)]$$

- Duality

$$\mathcal{F}[X(t)] = x(-f)$$

- Time shift:

$$\mathcal{F}[x(t - t_0)] = e^{-j2\pi ft_0}X(f)$$

- Scaling

$$\mathcal{F}[x(at)] = \frac{1}{|a|}X\left(\frac{f}{a}\right), \qquad a \neq 0$$

Fourier transform properties

- Modulation

$$\begin{cases} \mathcal{F}[e^{j2\pi f_0 t}x(t)] = X(f - f_0) \\ \mathcal{F}[x(t)\cos(2\pi f_0 t)] = \frac{1}{2}[X(f - f_0) + X(f + f_0)] \end{cases}$$

- Differentiation

$$\mathcal{F}[x'(t)] = j2\pi f X(f)$$

$$\mathcal{F}\left[\frac{d^n}{dt^n}x(t)\right] = (j2\pi f)^n X(f)$$

- Convolution

$$\mathcal{F}[x(t) \star y(t)] = X(f)Y(f)$$

$$\mathcal{F}[x(t)y(t)] = X(f) \star Y(f)$$

Fourier transform table

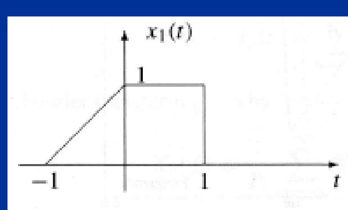
$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f t_0}$	$\delta(f - f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
$\Pi(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \quad \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \quad \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$

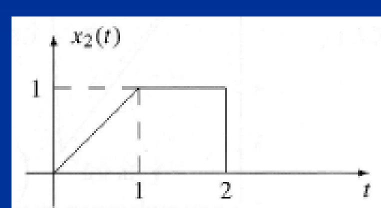
Fourier transform table

$e^{-\alpha\|t\|}, \quad \alpha > 0$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_0})$

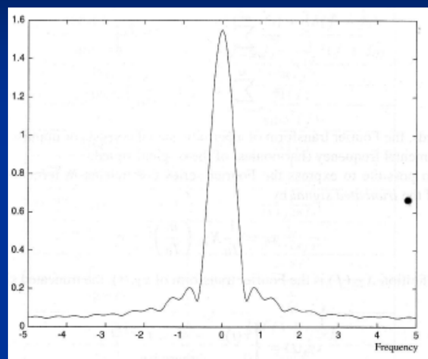
Fourier transform example

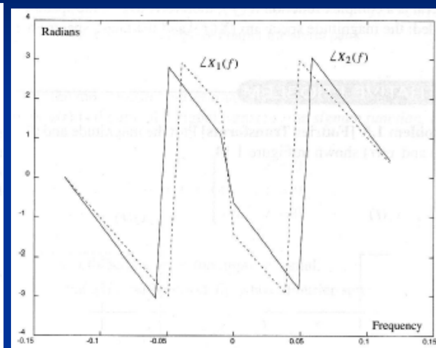
- Plot the magnitude and the phase spectra of signals x1(t) and x2(t)





Fourier transform example





|X1(f)| = |X2(f)|

Frequency-Domain Analysis of LTI System

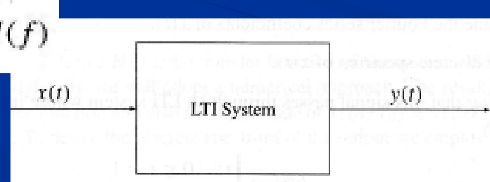
- The output of an LTI system with impulse response h(t) when the input signal is x(t) is given by the convolution integral

$$y(t) = x(t) \star h(t)$$

$$H(f) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t}\, dt$$

$$Y(f) = X(f)H(f)$$

$$\begin{cases} |Y(f)| = |X(f)|\,|H(f)| \\ \angle Y(f) = \angle X(f) + \angle H(f) \end{cases}$$



Power and Energy

$$\begin{cases} E_X = \int_{-\infty}^{\infty} x^2(t)\, dt \\ P_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \end{cases}$$

- A signal with finite energy is called an energy-type signal, and a signal with positive and finitepowerisapower-typesignal.
- x(t)= II(t)is an example of an energy-type signal, whereas x(t) = cos(t) is an example of a power-type signal.
- All periodic signals are power-type signals.

Energy and Power Signals

- Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$

- Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\, dt \quad \text{joules}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\, dt \quad \text{watts}$$

Energy and Power Signals

- For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- The normalized average power P of x(t) is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \qquad P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Energy and Power Signals

- x(t) (or x[n]) is said to be an energy signal (or sequence) if and only if 0 < E < oo, and so P = 0. For example, a signal having only one square pulse is energy signal. A signal that decays exponentially has finite energy, so, it is also an energy signal. The power of an energy signal is 0, because of dividing finite energy by infinite time (or length).
- x(t) (or x[n]) is said to be a power signal (or sequence) if and only if 0 < P < oo, thus implying that E = oo. For example, sine wave in infinite length is power signal.
- Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Power and Energy

- The energy spectral density of an energy-type signal gives the distribution of energy at various frequencies of the signal

$$\mathcal{G}_X(f) = |X(f)|^2$$

$$\mathcal{G}_X(f) = \mathcal{F}[R_X(\tau)]$$

$$E_X = \int_{-\infty}^{\infty} \mathcal{G}_X(f)\,df$$

Power and Energy

- the autocorrelation function of x (t), defined as

$$R_X(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt$$
$$= x(\tau) \star x(-\tau)$$

- Time-average autocorrelation function

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

Power and Energy

- The power-spectral density is in general given by

$$\mathcal{S}_X(f) = \mathcal{F}[R_X(\tau)]$$

- The total power is the integral of the power-spectral density given by

$$P_X = \int_{-\infty}^{\infty} \mathcal{S}_X(f)\,df$$